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Theory of q-Deformed Forms. III. q-Deformed Hodge Star, Inner Product, Adjoint Operator of Exterior Derivative, and Self-Dual Yang–Mills Equation

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In this paper we introduce the q-deformed Hodge star operator, q-deformed inner product, and q-deformed adjoint of the q-deformed exterior derivative and investigate their properties. Using this mathematical background, we construct the q-deformed self-dual Yang–Mills theory.

1. INTRODUCTION

Quantum groups provide a concrete example of noncommutative differential geometry (Connes, 1986). The idea of the quantum plane was first introduced by Manin (1988, 1989). The application of noncommutative differential geometry to quantum matrix groups was made by Woronowicz (1987, 1989). Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's quantum plane. They developed a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$, so-called $GL_q(n)$. Much subsequent work has been done in this direction (Schmidke *et al.*, 1989; Schirrmacher, 1991a,b; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Ubriaco, 1992; Giler *et al.*, 1991, 1992; Lukierski *et al.*, 1991; Lukierski and Nowicki, 1992; Castellani, 1992; Chaichian and Demichev, 1992; Chung, n.d.-a,b).

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In part I (Chung *et al.*, 1996) of this series of papers we proved associativity of the q-deformed wedge product and showed that the q-deformed wedge product satisfies a particular commutation relation.

In part II (Chung, 1996) we introduced the q-deformed differential forms and quantum-algebra-valued q-deformed forms. We used these to obtain the q-inner derivative and discussed its properties. We used these results to discuss the q-deformed Hamilton equation.

In this paper (part III) we introduce the q-deformed Hodge star operator, q-deformed inner product, and q-deformed adjoint of the q-deformed exterior derivative and investigate their properties. Using this mathematical background we construct the q-deformed self-dual Yang–Mills theory.

2. q-DEFORMED HODGE STAR

In this section we introduce the q-deformed Hodge star operator and investigate its properties. Throughout this paper we adopt the conventions and notations given in parts I and II. From part II we have the following commutation relations between the q-deformed differential forms:

$$dx^i \wedge_q dx^j = (-q)\, dx^j \wedge_q dx^i \qquad (i > j)$$
$$dx^i \wedge_q dx^i = 0 \qquad (i, j = 1, 2, \ldots, N) \tag{1}$$

where the q-deformed wedge product $\wedge_q$ reduces to the usual wedge product when q goes to 1. The relation (1) can be written in the form

$$dx^i \wedge_q dx^j = (-q)^{P(ij)} dx^j \wedge_q dx^i \tag{2}$$

where the symbol $P(ij)$ is defined as

$$P(ij) = 1 \qquad (i > j)$$
$$P(ij) = 0 \qquad (i = j)$$
$$P(ij) = -1 \qquad (i < j)$$

Now we introduce the Hodge star operator (*) as follows

$$*dx^I = \frac{1}{[N-p]!} E_{IJ}(q^{-2})\, dx^J \tag{3}$$

where $E_{IJ}(q^{-2})$ denotes replacing q with q^{-2} in the q-deformed Levi-Civita symbol defined in parts I and II and I and J denote $i_1, i_2, \ldots, i_p$ and $j_1, j_2, \ldots, j_{N-p}$, respectively. Now we show that the Hodge star operator satisfies the following properties:

$$**dx^I = (-q^{-2})^{p(N-p)} dx^I \tag{4}$$

where dx^J is a q-deformed p-form. In obtaining (4) we used the following properties of the q-deformed Levi-Civita symbol:

$$\sum_{J, j_1 < j_2 < \cdots < j_{N-p}} E_{IJ}(q^{-2}) E_{JK}(q^{-2}) = (-q^{-2})^{p(N-p)} \delta^{i_1}_{[k_1} \delta^{i_2}_{k_2} \cdots \delta^{i_p}_{k_p]_{q^{-2}}} \quad (5)$$

where the q-symmetrizer is defined in Chung *et al.* (1994). The q-symmetrizer satisfies the following property:

$$\sum_{K} \delta^{i_1}_{[k_1} \delta^{i_2}_{k_2} \cdots \delta^{i_p}_{k_p]_{q^{-2}}} \, dx^K = [p]! \, dx^I \quad (6)$$

The general proof is given in the Appendix and here we show that (4) holds for the $N = 4$ and $p = 2$ case. Let us consider the two-form

$$dx^i \wedge_q dx^j \qquad (i < j) \quad (7)$$

Then we have

$$\begin{aligned} *(dx^i \wedge_q dx^j) &= \frac{1}{[2]!} E_{ijkl}(q^{-2}) \, dx^k \wedge_q dx^l \\ &= \sum_{k<l} E_{ijkl}(q^{-2}) \, dx^k \wedge_q dx^l \end{aligned}$$

Applying the Hodge star operation to the two-form (7) twice, we find

$$\begin{aligned} **(dx^i \wedge_q dx^j) &= \sum_{k<l} E_{ijkl}(q^{-2}) *(dx^k \wedge_q dx^l) \\ &= \frac{1}{[2]!} \sum_{k<l} E_{ijkl}(q^{-2}) E_{klmn}(q^{-2}) \, dx^m \wedge_q dx^n \\ &= \frac{1}{[2]!} (-q^{-2})^4 (\delta^i_m \delta^j_n - q^{-2} \delta^i_n \delta^j_m) \, dx^m \wedge_q dx^n \\ &= (q^{-2})^4 x^i \wedge_q dx^j \end{aligned}$$

where we used the properties

$$\sum_{k<l} E_{ijkl}(q^{-2}) E_{klmn}(q^{-2}) = (-q^{-2})^4 (\delta^i_m \delta^j_n - q^{-2} \delta^i_n \delta^j_m) \qquad \text{for} \quad i < j$$

and

$$dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_N} = E_{i_1 \cdots i_N}(q^{-2}) \, dx^1 \wedge_q \cdots dx^N \quad (8)$$

3. q-DEFORMED INNER PRODUCT

In this section we discuss the q-deformed inner product (scalar product). Consider the following two q-deformed p-forms α_p and β_p:

$$\alpha_p = \sum_{i_1<\cdots<i_p} \alpha_{i_1\cdots i_p}\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p}$$

$$\beta_p = \sum_{j_1<\cdots<j_p} \beta_{j_1\cdots j_p}\, dx^{j_1} \wedge_q \cdots \wedge_q dx^{j_p}$$

Then we define the q-deformed inner product as the integral

$$(\alpha_p, \beta_p) = \int \alpha_p \wedge_q^* \beta_p \tag{9}$$

where $*\beta_p$ is given by

$$*\beta_p = \sum_{j_1<\cdots<j_p, j_{p+1}<\cdots<j_N} \beta_{j_1\cdots j_p} E_{j_1\cdots j_p j_{p+1}\cdots j_N}(q^{-2})\, dx^{j_{p+1}} \wedge_q \cdots \wedge_q dx^{j_N}$$

Computing the q-deformed inner product, we obtain

$$(\alpha_p, \beta_p) = \int (-q^{-2})^{2(\Sigma_{k=1}^{p} i_k - 6)} \alpha_{i_1\cdots i_p} \beta_{i_1\cdots i_p} dx^1 \wedge_q \cdots \wedge_q dx^N \tag{10}$$

The proof is easy. We have

$$\begin{aligned}(\alpha_p, \beta_p) = \int & \alpha_{i_1\cdots i_p}\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_p} \wedge_q \\ & \beta_{j_1\cdots j_p} E_{j_1\cdots j_p j_{p+1}\cdots j_N}(q^{-2}) dx^{j_{p+1}} \wedge_q \cdots \wedge_q dx^{j_N} \\ = \int & \alpha_{i_1\cdots i_p} \beta_{j_1\cdots j_p} E_{j_1\cdots j_p j_{p+1}\cdots j_N}(q^{-2})\, dx^{j_{p+1}} \\ & \times E_{i_1\cdots i_p j_{p+1}\cdots j_N}(q^{-2})\, dx^{j_{p+1}}\, dx^1 \wedge_q \cdots \wedge_q dx^N \\ = \int & (-q^{-2})^{2(\Sigma_{k=1}^{p} i_k - 6)} \alpha_{i_1\cdots i_p} \beta_{i_1\cdots i_p} dx^1 \wedge_q \cdots \wedge_q dx^N\end{aligned}$$

where we used the contraction rule for the q-deformed Levi-Civita symbol,

$$E_{j_1\cdots j_p j_{p+1}\cdots j_N}(q^{-2}) E_{i_1\cdots i_p j_{p+1}\cdots j_N}(q^{-2}) = (-q^{-2})^{2(\Sigma_{k=1}^{p} i_k - 6)} \delta^{j_1}_{[i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_N}_{i_N]_{q^{-2}}} \tag{11}$$

The q-deformed inner product has the further property that

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p) \tag{12}$$

because of the identity

$$\alpha_p \wedge_q^* \beta_p = \beta_p \wedge_q^* \alpha_p \tag{13}$$

4. q-DEFORMED ADJOINT OF q-DEFORMED EXTERIOR DERIVATIVE

In this section we discuss the q-deformed adjoint of the q-deformed exterior derivative. We define the q-deformed adjoint of the q-deformed exterior derivative d as

$$\delta = -(-q^{-2})^{-(N-p+2)(p-1)*}d* \tag{14}$$

The q-deformed adjoint has the property

$$(\delta\alpha_p, \beta_{p-1}) = (\alpha_p, d\beta_{p-1}) \tag{15}$$

We will prove the identity (15) as follows. By definition we find

$$\begin{aligned}
(\delta\alpha_p, \beta_{p-1}) &= (\beta_{p-1}, \delta\alpha_p) \\
&= \int \beta_{p-1} \wedge_q^* \delta\alpha_p \\
&= -(-q^{-2})^{-(N-p+2)(p-1)} \int \beta_{p-1} \wedge_q^{**} d^*\alpha_p \\
&= -(-q^{-2})^{-(N-p+2)(p-1)}(-q^{-2})^{(N-p+1)(p-1)} \int \beta_{p-1} \wedge_q d^*\alpha_p \\
&= -(-q^{-2})^{-(p-1)} \int \beta_{p-1} \wedge_q d^*\alpha_p \\
&= -(-q^{-2})^{-(p-1)} \int \beta_{i_1\cdots i_{p-1}} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_{p-1}} \\
&\quad \times d^*(\alpha_{j_1\cdots j_p} dx^{j_1} \wedge_q \cdots \wedge_q dx^{j_p})
\end{aligned}$$

where we note that

$$i_1 < \cdots < i_{p-1}, \qquad j_1 < \cdots < j_p$$

Then the right-hand side is

$$\begin{aligned}
&= -(-q^{-2})^{-(p-1)} \int \beta_{i_1\cdots i_{p-1}} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_{p-1}} \\
&\quad \times d(\alpha_{j_1\cdots j_p} E_{j_1\cdots j_p j_{p+1}\cdots j_N} dx^{j_{p+1}} \wedge_q \cdots \wedge_q dx^{j_N})
\end{aligned}$$

$$= -(-q^{-2})^{-(p-1)} \int \beta_{i_1 \cdots i_{p-1}} \partial_a \alpha_{j_1 \cdots j_p} E_{j_1 \cdots j_p j_{p+1} \cdots j_N}$$
$$\times\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_{p-1}} \wedge_q dx^a \wedge_q dx^{j_{p+1}} \wedge_q \cdots \wedge_q dx^{j_N})$$
$$= -(-q^{-2})^{-(p-1)} \int \beta_{i_1 \cdots i_{p-1}} \partial_a \alpha_{j_1 \cdots j_p} E_{j_1 \cdots j_p j_{p+1} \cdots j_N} E_{i_1 \cdots i_{p-1} a j_{p+1} \cdots j_N}$$
$$\times\, dx^1 \wedge_q \cdots \wedge_q dx^N$$
$$= -(-q^{-2})^{-(p-1)} \int \beta_{i_1 \cdots i_{p-1}} \partial_a \alpha_{j_1 \cdots j_p} \delta^{j_1}_{i_1} \cdots \delta^{j_{p-1}}_{i_{p-1}} \delta^{j_p}_a$$
$$\times\, (-q^{-2})^{2(\Sigma_{k=1}^{p} j_k - 6)} dx^1 e \cdots \wedge_q dx^N$$
$$= -(-q^{-2})^{-(p-1)} \int \beta_{i_1 \cdots i_{p-1}} \partial_a \alpha_{i_1 \cdots i_{p-1} a} (-q^{-2})^{2(\Sigma_{k=1}^{p-1} i_k + a - 6)} dx^1 \wedge_q \cdots \wedge_q dx^N$$
$$= (-q^{-2})^{-(p-1)}$$
$$\times \int \alpha_{i_1 \cdots i_{p-1} a} \partial_a \beta_{i_1 \cdots i_{p-1}} (-q^{-2})^{2(\Sigma_{k=1}^{p-1} i_k + a - 6)} dx^1 \wedge_q \cdots \wedge_q dx^N$$
$$= \int \alpha_{a i_1 \cdots i_{p-1}} \partial_a \beta_{i_1 \cdots i_{p-1}} (-q^{-2})^{2(\Sigma_{k=1}^{p-1} i_k + a - 6)} dx^1 \wedge_q \cdots \wedge_q dx^N$$
$$= \int \alpha_p \wedge_q^* d\beta_p$$
$$= (\alpha_p, d\beta_p)$$

which completes the proof of the identity (15).

5. q-DEFORMED SELF-DUAL YANG–MILLS EQUATION

In this section we use the formulas given in part II to discuss the q-deformed Maxwell and Yang–Mills theory. Let us define the q-vector 1-form $A \in \Lambda_q^1(V)$,

$$A = \sum_i A_i \, dx^i \tag{16}$$

The q-deformed field strength 2-form F is defined by acting with the q-exterior derivative d,

$$F = dA \tag{17}$$

$$= \sum_{i<j} (\partial_i A_j + E^{ij}_{ji} \partial_j A_i)\, dx^i \wedge_q dx^j$$

$$= \sum_{i>j} (\partial_i A_j + E^{ij}_{ji} \partial_j A_i)\, dx^i \wedge_q dx^j \tag{18}$$

If we define F as

$$F = \sum_{i<j} F_{ij}\, dx^i \wedge_q dx^j$$

$$= \sum_{i>j} F_{ij}\, dx^i \wedge_q dx^j$$

$$= \frac{1}{2} \sum_{i,j} F_{ij}\, dx^i \wedge_q dx^j \tag{19}$$

then we have

$$F_{ij} = \partial_i A_j + E^{ij}_{ji} \partial_j A_i \tag{20}$$

which implies that

$$F_{ij} = \partial_i A_j - q^{-1} \partial_j A_i \qquad (i < j) \tag{21}$$

$$F_{ij} = \partial_i A_j - q \partial_j A_i \qquad (i > j) \tag{22}$$

Therefore we have

$$F_{ij} = -q^{-1} F_{ji} \qquad (i < j) \tag{23}$$

If we define the dual tensor $*F_{ij}$ as

$$*F_{ij} = \sum_{k<l} E_{klij}(q^{-2}) F_{ij} \qquad (i < j) \tag{24}$$

then we have the following q-deformed self-dual condition:

$$*F = \tilde{F} \tag{25}$$

where $\tilde{F}$ is defined as

$$\tilde{F} = \sum_{i<j} \sum_{k<l} E_{klij}(q^{-2}) F_{ij}\, dx^i \wedge_q dx^j \tag{26}$$

These can be written for each component as follows:

$$*F_{12} = (-q^{-2})^4 F_{12} \tag{27}$$

$$*F_{13} = (-q^{-2})^3 F_{13} \tag{28}$$

$$*F_{14} = (-q^{-2})^2 F_{14} \tag{29}$$

$$*F_{23} = (-q^{-2})^2 F_{23} \tag{30}$$

$$*F_{24} = (-q^{-2}) F_{24} \tag{31}$$

$$*F_{34} = F_{34} \tag{32}$$

which implies that

$$F_{12} = F_{34} \tag{33}$$

$$F_{13} = F_{24} \tag{34}$$

$$F_{23} = F_{14} \tag{35}$$

Now we generalize the q-deformed self-dual Maxwell equation to the q-deformed Yang–Mills equation. Let us introduce the quantum-algebra-valued q-deformed vector 1-form

$$A = A_i^a \, dx^i \, T_a \tag{36}$$

and the q-deformed field strength tensor 2-form

$$F = DA = dA + \tfrac{1}{2}[A, A]_q \tag{37}$$

If we define the q-deformed field strength tensor 2-form by

$$\begin{aligned} F &= \sum_{i<j} F_{ij}^a dx^i \wedge_q dx^j \, T_a \\ &= \sum_{i>j} F_{ij}^a dx^i \wedge_q dx^j \, T_a \\ &= \frac{1}{2} \sum_{i,j} F_{ij}^a dx^i \wedge_q dx^j \, T_a \end{aligned} \tag{38}$$

we have

$$\begin{aligned} F_{ij}^a &= \partial_i A_j^a - q^{-1} \partial_j A_i^a + \tfrac{1}{2} f_{abc} (A_i^b A_j^c + q^{-1} A_j^c A_i^b) \qquad (i < j) \\ F_{ij}^a &= \partial_i A_j^a - q \partial_j A_i^a + \tfrac{1}{2} f_{abc} (A_i^b A_j^c + q A_j^c A_i^b) \qquad (i > j) \end{aligned} \tag{39}$$

If we demand that the commutation relation between A_j^c and A_i^b is given by

$$\begin{aligned} A_j^c A_i^b &= q^{-1} A_i^b A_j^c \qquad (j < i) \\ A_j^c A_i^b &= q A_i^b A_j^c \qquad (j > i) \end{aligned} \tag{40}$$

then the q-deformed field strength tensor is given by

$$\begin{aligned} F_{ij}^a &= \partial_i A_j^a - q^{-1} \partial_j A_i^a + f_{abc} A_i^b A_j^c \qquad (i < j) \\ F_{ij}^a &= \partial_i A_j^a - q \partial_j A_i^a + f_{abc} A_i^b A_j^c \qquad (i > j) \end{aligned} \tag{41}$$

Then the q-deformed self-dual condition is obtained in analogy with (25) as follows:

$$*F^a_{ij} = \sum_{k<l} E_{klij}(q^{-2})F^a_{ij} \qquad (i < j) \tag{42}$$

$$*F = \tilde{F} \tag{43}$$

where $\tilde{F}$ is defined as

$$\tilde{F} = \sum_{i<j}\sum_{k<l} E_{klij}(q^{-2})F^a_{ij}\, dx^i \wedge_q dx^j\, T_a \tag{44}$$

6. CONCLUSIONS

In this paper we have used the q-deformed differential forms and quantum-algebra-valued q-deformed forms given in part II to obtain a q-deformed Hodge-dual operator, q-deformed inner product, and q-deformed adjoint of the q-deformed exterior derivative and have discussed their properties. As a physical application, we have discussed the q-deformed self-dual Maxwell equation and Yang–Mills equation. In this case we find that the ordinary self-dual condition should be q-deformed in a more complicated form. We think that much will be accomplished in this direction. In particular we hope that the q-deformed Lagrangian equation of motion of the q-deformed mechanics and q-deformed Maxwell and Yang–Mills theory will be clarified in the near future.

APPENDIX. PROOF OF EQUATION (4)

By definition we have

$$\begin{aligned} *(dx^I) &= \frac{1}{[N-p]!}\sum_J E_{IJ}(q^{-2})\, dx^J \\ &= \sum_{J,\text{ordered}} E_{IJ}(q^{-2})\, dx^J \end{aligned}$$

Acting with the Hodge star operator again, we find

$$\begin{aligned} **(dx^I) &= \sum_{J,\text{ordered}} E_{IJ}(q^{-2}) \sum_K \frac{1}{[p]!} E_{JK}E_{JK}(q^{-2})\, dx^K \\ &= \sum_K \frac{1}{[p]!}(-q^{-2})^{p(N-p)}\, \delta^{i_1}_{[k_1}\delta^{i_2}_{k_2}\cdots\delta^{i_p}_{k_p]_{q^{-2}}}\, dx^K \\ &= (-q^{-2})^{p(N-p)} dx^I \end{aligned}$$

which completes the proof of (4).

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